Rectangular *L*-matrices

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ABSTRACT

An *L-matrix* is an m-by-n $(0, 1, -1)$-matrix A such that every m-by-n real matrix with the same sign pattern as A has linearly independent rows. The most widely studied L-matrices are the square L-matrices, also called sign-nonsingular matrices. In this paper we investigate rectangular L-matrices. We obtain and study a decomposition theorem for L-matrices. We introduce two new classes of L-matrices, which for square matrices reduce to sign-nonsingular matrices. The maximum number of columns for matrices in each of these classes is obtained, and those matrices attaining the maximum are characterized.

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1. INTRODUCTION

The study of the sign solvability of linear systems of equations has been shown [5] to reduce to the study of two classes of matrices. The first is the class of m-by-$m+1$ matrices known as S-matrices (and the more general class of S^*-matrices). An m-by-$m+1$ real matrix A is an *S-matrix* provided that the right nullspace of A, and that of every matrix with the same sign pattern as A, is spanned by a vector with positive coordinates. An *S^*-matrix* is a matrix obtained from an S-matrix by multiplying some of its columns by -1. The class of S-matrices has a simple inductive structure, and there is a polynomial-time algorithm to recognize whether or not a matrix is an S-matrix [4,5]. The second class is the class of general rectangular m-by-n L-matrices and has a more complex structure. An m by n real matrix A is an *L-matrix* provided A, and every matrix with the same sign pattern as A, has linearly independent rows.[§] It been shown in [5] that the recognition problem for L-matrices is NP-complete. According to conventional wisdom, this suggests that no polynomial-time recognition algorithm exists for L-matrices. It also suggests that it is probably difficult to obtain good results for general L-matrices.

The most widely studied L-matrices are the square L-matrices [1,3,5,8]. A square L-matrix is also called a *sign-nonsingular matrix*, abbreviated SNS matrix. In contrast to the class of general L-matrices, the complexity of the recognition problem for SNS matrices is unknown. In addition, a significant body of facts is known about SNS matrices. An important tool in the study of SNS matrices is the determinant. This is because a square matrix is an SNS matrix if and only if there is a nonzero term in its determinant expansion and every nonzero term has the same sign (see e.g. [1,3]).

The determinant does not seem to be generally useful in the study of nonsquare L-matrices. We justify this statement as follows. Deleting $n-m$ rows of an SNS matrix of order n results in an L-matrix containing an SNS submatrix of order m. If an m-by-n matrix A contains an SNS submatrix of order m, then A is an L-matrix; in addition,

$$\begin{bmatrix} \multicolumn{2}{c}{AQ} \\ I_{n-m} & O \end{bmatrix}$$

is an SNS matrix for some permutation matrix Q and thus A is a submatrix of an SNS matrix of order n. However, there are m by n L-matrices which do not contain any SNS submatrix of order m. Let

$$S_3 = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}. \tag{1}$$

[§] In studying sign solvability, a matrix is usually called an L-matrix provided it and each matrix with the same sign pattern has linearly independent columns. For convenience, we have altered the definition and have replaced columns by rows.

The fact that S_3 does not contain an SNS submatrix of order 3 is a consequence of the well-known fact that an SNS matrix of order 3 contains a 0. For every nonzero diagonal $(0, 1, -1)$-matrix D of order 3, there is some column of DS_3 each of whose nonzero entries has the same sign. This implies that for every matrix $\widetilde{S}_3$ with the same sign pattern as S_3 and every nonzero diagonal matrix $\widetilde{D}$, the sum of the rows of $\widetilde{D}\widetilde{S}_3$ is not zero. Therefore S_3 is an L-matrix. We shall see later that for each $m \geqslant 2$, there exist m-by-n L-matrices for which the largest order of an SNS submatrix is 2.

The purpose of this paper is to demonstrate that there are certain subclasses of the rectangular L-matrices for which some interesting and, in some cases, surprising results can be obtained. In the next section we introduce these classes and then summarize our results. Later sections are concerned with the precise statement and verification of these results.

2. A CATALOGUE OF L-MATRICES

Since the properties of matrices that we study depend only on the signs of the entries and not on their magnitudes, without loss of generality we consider throughout matrices whose entries are 0, 1, and -1, that is, $(0, 1, -1)$-matrices. Thus an L-matrix is an m-by-n $(0, 1, -1)$-matrix A such that every m by n real matrix with the same sign pattern as A has linearly independent rows. Except for the results concerning the complexity of recognizing L-matrices, the only result that seems to be in the literature about general L-matrices is the following basic property [5]:

> (*) An m-by-n $(0, 1, -1)$-matrix A is an L-matrix if and only if for every diagonal $(0, 1, -1)$ matrix $D \neq O$ of order m, there exists a nonzero column of DA each of whose nonzero entries has the same sign.

The reason is that if there is a $D \neq O$ such that each column of DA either is zero or has entries of opposite signs, then there exists a matrix X with the same sign pattern as A having two disjoint sets of rows whose sums are equal, and hence A is not an L-matrix.

Property (*) motivates the introduction of the following concepts for real vectors. A real vector x is called

(i) *balanced* provided that either x is a zero vector or x has both a positive entry and a negative entry;
(ii) *unisigned* provided it is not balanced,
(iii) $(+, 0)$-*unisigned* provided $x \neq 0$ and x is unisigned with all nonzero entries of x positive,

(iv) $(-, 0)$-*unisigned* provided $x \neq 0$ and x is unisigned with all nonzero entries of x negative.

A diagonal $(0, 1, -1)$ matrix D is called

(i) a *signing* provided $D \neq O$, and
(ii) a *strict signing* provided there are no 0's on the main diagonal of D, that is, provided D is a nonsingular matrix.

The matrix DA is called

(i) a *signing of* A provided D is a signing,
(ii) a *strict signing of* A provided D is a strict signing, and
(iii) a *balanced (strict) signing* provided DA is a (strict) signing of A in which each column of DA is balanced.

Clearly, if D is a strict signing then A is an L-matrix if and only if DA is an L-matrix. Similarly, if E is a nonsingular diagonal $(0, 1, -1)$ matrix of order n, then A is an L-matrix if and only if AE is an L-matrix. Property $(*)$ can be reformulated as:

> $(*)$ A $(0, 1, -1)$ matrix is not an L-matrix if and only if it has a balanced signing.

We introduce and study in this paper two classes of rectangular L-matrices which we call totally L-matrices and barely L-matrices (defined below). As the names suggest, totally L-matrices are L-matrices in a very strong sense, and barely L-matrices are L-matrices which are almost not L-matrices. Totally L-matrices generalize the SNS matrices and the S^*-matrices defined in section 1. Barely L-matrices can be also be regarded as generalizations of SNS matrices.

In contrast to the 3-by-4 L-matrix S_3 in (1), which has no SNS submatrix of order 3, every submatrix of order 2 of the 2-by-4 L-matrix

$$T = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \tag{2}$$

is an SNS matrix. We define an m-by-n $(0, 1, -1)$ matrix A to be a *totally L-matrix* provided *every submatrix of A of order m is an SNS matrix.* The class of totally L-matrices is contained in the class of L-matrices and contains the classes of SNS matrices and S^*-matrices. Also, every 1-by-n matrix with no 0's is a totally L-matrix. It follows easily that an m-by-n $(0, 1, -1)$ matrix A is a totally L-matrix if and only if every nonzero vector in the null spaces of the matrices with the same sign pattern as A has at least $m + 1$ nonzero coordinates.

Let A be an m-by-$m+1$ totally L-matrix, and let X be a matrix with the same sign pattern as A. Then the rank of X equals m, and thus the nullspace of X has dimension 1. Let $X_{(i)}$ ($i = 1, 2, \ldots, m+1$) denote the matrix of order m obtained by deleting column i of X. Then by Cramer's rule the nullspace is spanned by the vector

$$z_X = \begin{bmatrix} -\det X_{(1)} \\ \vdots \\ (-1)^i \det X_{(i)} \\ \vdots \\ (-1)^{m+1} \det X_{(m+1)} \end{bmatrix}.$$

Since each matrix $A_{(i)}$ ($i = 1, 2, \ldots, m+1$) is an SNS matrix, the sign of $\det X_{(i)}$ equals the sign of $\det A_{(i)}$. Hence if Y also has the same sign pattern as A, then z_Y and z_X have the same sign pattern. Let D be the signing of order $m+1$ whose ith diagonal entry equals the sign of the ith entry of z_X ($i = 1, 2, \ldots, m+1$). Then AD is a totally L-matrix for which the vectors in the null spaces either have all positive coordinates or all have negative coordinates. Hence AD is an S-matrix and A is an S^*-matrix. Thus m-by-$m+1$ totally L-matrices are the same as S^*-matrices, and up to multiplication of some columns by -1, m-by-$m+1$ totally L-matrices are S-matrices.

A matrix obtained from an L-matrix by including additional columns is also an L-matrix. Thus it is natural to consider the subclass of L-matrices for which every column is essential (in order that the matrix be an L-matrix). Similarly, a matrix obtained from an L-matrix by deleting rows is an L-matrix. It is thus also natural to consider the subclass of L-matrices which cannot be enlarged by the inclusion of more rows. An m-by-n L-matrix is

(i) a *barely L-matrix* provided that each of its m-by-$n-1$ submatrices is not an L-matrix, and
(ii) an *extremal L-matrix* provided no $m+1$-by-n matrix containing A is an L-matrix.

Every m-by-n L-matrix contains an m-by-n' barely L-submatrix for some $n' \leqslant n$. Clearly an SNS matrix is a barely L-matrix. (Thus both totally L-matrices and barely L-matrices generalize SNS matrices.) The matrix S_3 in (1) is a 3-by-4 barely L-matrix. Any SNS matrix and the matrix S_3 are extremal L-matrices. In fact, we show in Section 5 that a matrix is an extremal L-matrix if and only if it is a barely L-matrix. In Section 3 we introduce the concept of an L-indecomposable matrix (one that cannot be written as a subdirect sum of two smaller L-matrices) and obtain a decomposition of an L-matrix into L-indecomposable components.

We now summarize the main results established in this paper:

Theorem 1: An L-matrix is an L-indecomposable, barely L-matrix if and only if for each column i there is a strict signing DA of A whose only unsigned column is column i.

Theorem 6: An m by n totally L-matrix with $m \geqslant 2$ satisfies $n \leqslant m+2$.

Theorem 11: A recursive characterization of totally L-matrices is given in terms of two operations called single extension and double extension. Thus for $m \geqslant 2$, there are three classes of m-by-n totally L-matrices: SNS matrices (for which no polynomial recognition algorithm is known[4]), S^*-matrices (for which a recognition algorithm with time complexity $O(m^2)$ is known[3,4]), and totally L-matrices with $n = m+2$ (for which a polynomial recognition algorithm of time complexity $O(m)$ follows from our recursive structure).

Theorem 16: Barely L-matrices are the same as extremal L-matrices.

Theorem 17: In contrast to general L-matrices, a barely L-matrix has a unique set of L-indecomposable components.

Theorem 18: An m-by-n barely L-matrix A satisfies $m \leqslant 2^{m-1}$ with equality for $m \geqslant 3$ if and only if the columns of A consist of exactly one of each m-tuple of 1's and -1's and its negative (S_3 is an instance).

Theorem 23: An m-by-n barely L-matrix of 0's and 1's (no -1's) with $m > 2$ satisfies

$$n \leqslant \binom{m}{\lceil \frac{m+1}{2} \rceil}$$

with equality characterized.

3. L-MATRICES

Let X_1 and X_2 be $(0, 1, -1)$-matrices. If X_1 and X_2 are L-matrices, then it follows from the definition that for all choices of Y of appropriate size the matrix

$$A = \begin{bmatrix} X_1 & O \\ Y & X_2 \end{bmatrix}$$

is an L-matrix. If the matrix A is an L-matrix, then the matrix X_1 is also an L-matrix but this need not be true of X_2. For example, if the matrix

$$\begin{bmatrix} X_1 \\ Y \end{bmatrix}$$

is an L-matrix then the matrix A is an L-matrix for all choices of X_2. However, if X_1 is a barely L-matrix, we show in section 5 that X_2 is an L-matrix.

The above remarks motivate the following definition. Let A be an arbitrary L-matrix. Then A is called *L-decomposable* provided there exist permutation matrices P and Q such that

$$A = P\begin{bmatrix} X_1 & O \\ Y & X_2 \end{bmatrix}Q \tag{3}$$

where X_1 and X_2 are (nonvacuous) L-matrices. We call (3) an *L-decomposition* of A. If A is not L-decomposable, then A is called *L-indecomposable*. It follows by induction that there exist permutation matrices R and S and an integer $k \geqslant 1$ such that

$$A = R\begin{bmatrix} A_1 & O & \cdots & O \\ A_{21} & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_k \end{bmatrix}S, \tag{4}$$

where $A_1, A_2, \ldots, A_k$ are L-indecomposable L-matrices. We call $\{A_1, A_2, \ldots, A_k\}$ a *set of L-indecomposable components* of A. The L-indecomposable components of an L-matrix are not in general unique. For example, let

$$A = \left[\begin{array}{ccc|cc} 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

Then A is L-decomposed with two L-indecomposable components of sizes 2 by 3 and 2 by 2, respectively. Permuting, we obtain

$$\left[\begin{array}{cc|c|cc} 1 & 1 & 0 & 0 & 0 \\ \hline 1 & -1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right],$$

which is an L-decomposition with three L-indecomposable components of sizes 1 by 2, 1 by 1, and 2 by 2, respectively. However, in section 5 we show that a barely L-matrix has, up to row and column permutations, a unique set of L-indecomposable components.

We now characterize L-indecomposable barely L–matrices in terms of signings. Let A be an m-by-n $(0, 1, -1)$-matrix. We denote the set of signings D such that column i is the unique unisigned column of DA by $\mathcal{A}_i$, $(1 \leqslant i \leqslant n)$. It follows that $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for $i \neq j$ and that an L-matrix A is a barely L-matrix if and only if $\mathcal{A}_i \neq \emptyset$ for each i.

THEOREM 1. *Let A be an m by n L-matrix. Then A is an L-indecomposable, barely L-matrix if and only if for each integer j with $1 \leqslant j \leqslant n$, $\mathscr{A}_j$ contains a strict signing.*

Proof. First suppose that A is an L-indecomposable barely L-matrix. Let j be an integer with $1 \leqslant j \leqslant n$, and let D be a signing in $\mathscr{A}_j$ with the maximum number k of nonzero entries. Assume to the contrary that $k < m$. Without loss of generality we assume that $D = D_1 \oplus D_2$ where D_1 is a strict signing and $D_2 = O$, and that A has the form

$$\begin{bmatrix} X & O \\ Y & Z \end{bmatrix},$$

where X has k rows and each column of X contains a nonzero entry. Suppose that Z is vacuous. Then each column of A contains a nonzero in its first k, rows and hence each column other than column i of DA contains both a 1 and a -1 in its first k rows. It now follows that there exists a strict signing $D' = D_1 \oplus D_2'$ for some D_2' such that $D' \in \mathscr{A}_i$, contrary to the choice of D. We thus conclude that Z is a nonvacuous matrix. Since A is L-indecomposable, Z is not an L-matrix. Thus there exists a signing E such that each column of EZ is balanced. Since A is an L-matrix, some column of $(D_1 \oplus E)A$ is unisigned. Since every column of D_1X other than column i contains both a 1 and a -1 and each column of EZ is balanced, column i is the only unisigned column of $(D_1 \oplus E)A$. Hence $D_1 \oplus E$ is a signing in $\mathscr{A}_i$ with more than k nonzero entries, contradicting our choice of D. Therefore, each A_j contains a strict signing.

Now suppose that for each integer j with $1 \leqslant j \leqslant n$, $\mathscr{A}_j$ contains a strict signing. Then since A is an L-matrix and each A_j is nonempty, A is a barely L-matrix. If

$$A = \begin{bmatrix} X & O \\ Y & Z \end{bmatrix}$$

where Z is and L-matrix, then no strict signing of A has only its first column unisigned. It follows that A is L-indecomposable. ∎

A square matrix A of order n is called *partly decomposable* [2] provided there exist permutation matrices P and Q and an integer k with $1 \leqslant k \leqslant n-1$ such that PAQ has the form

$$\begin{bmatrix} U & O \\ V & W \end{bmatrix} \tag{5}$$

where U is a square matrix of order k. The matrix A is *fully indecomposable* provided it is not partly decomposable. Full indecomposability is a purely combinatorial property of a square matrix. However for SNS matrices it is equivalent to L-indecomposability.

THEOREM 2. *Let A be an SNS matrix. Then A is L-indecomposable if and only if A is fully indecomposable.*

Proof. First suppose that A is partly decomposable. Then the square matrices U and W in (5) are SNS matrices and hence A is L-decomposable. Now suppose that A is L-decomposable. Then the L-matrices X_1 and X_2 in (3) must be square, and hence A is partly decomposable. ∎

The following theorem is an immediate consequence of Theorems 1 and 2 and is contained in [7].

THEOREM 3. *Let A be a fully indecomposable* $(0, 1, -1)$ *matrix. Then A is an SNS matrix if and only if every strict signing of A contains a unisigned column.*

4. TOTALLY L-MATRICES

Let $A = [a_{ij}]$ be an m-by-n matrix. We denote by $A(i, j)$ the submatrix of A obtained by deleting row i and column j. The submatrix of A obtained by deleting column j is denoted by $A(j)$. If $1 \leqslant i_1 < \cdots < i_p \leqslant m$ and $1 \leqslant j_1 < \cdots < j_q$, then $A[i_1, \dots, i_p | j_1, \dots, j_q]$ is the p by q submatrix of A determined by rows $i_1, \dots, i_p$ and columns $j_1, \dots, j_q$. The matrix of all 1's of order n is denoted by J_n.

LEMMA 4. *Let A be an m-by-$m+1$ L-matrix. Let $x = (x_1, x_2, \dots, x_{m+1})$ be the vector in which*

$$x_j = \begin{cases} 0 & \text{if } A(j) \text{ is not an SNS matrix,} \\ (-1)^{m+1+j} \operatorname{sign}\det A(j) & \text{if } A(j) \text{ is an SNS matrix.} \end{cases}$$

If A has at least one SNS submatrix of order m, then the matrix

$$\left[\begin{array}{c} A \\ \hline x \end{array}\right]$$

is an SNS matrix of order $m+1$.

Proof. The lemma is a consequence of the fact that a $(0, 1, -1)$ matrix is an SNS matrix if and only if there is a nonzero term in its determinant expansion and every nonzero term has the same sign. ∎

It is well known (see e.g. [2]) that for $n \geqslant 2$, a matrix A of order n is fully indecomposable if and only if there is a nonzero term in the determinant expansion of each submatrix $A(i, j)$. This implies, in particular, that if A is a fully indecomposable SNS matrix and $a_{ij} \neq 0$, then $A(i, j)$ is an SNS matrix.

A column of a matrix is a *unit column* provided it has exactly one nonzero entry.

THEOREM 5. *Let*

$$A = [\ F \mid b\ ],$$

be an m-by-m + 1 totally L-matrix. If F is a fully indecomposable matrix of order m, then b is a unit column.

Proof. Since A is a totally L-matrix, it follows from Lemma 4 that there is a $(1, -1)$ vector x such that the matrix

$$B = \left[\frac{A}{x}\right]$$

is an SNS matrix. Now assume that F is fully indecomposable. Without loss of generality we may assume that x is a vector of all 1's (by adjusting the signs of the columns of A) and that b is a $(0, 1)$ vector (by adjusting the signs of the rows of A). Suppose that there is a 1 in row i of b and that there is a 1 in row i and column k of F for some k. The submatrix $F(i, k)$ of F has a nonzero term in its determinant expansion, and its complementary submatrix $B[i, m+1|k, m+1]$ in B equals J_2. Since J_2 is not an SNS matrix, this contradicts the sign-nonsingularity of B. Hence whenever b has a 1 in a row, then each entry of F in that row equals 0 or -1.

Suppose to the contrary that b contains at least two 1's, which we may assume are in its first two rows. Suppose that some column, say column 1, of F contains a -1 in both row 1 and row 2. Since $F(1, 1)$ has a nonzero term in its determinant expansion and since $H = B[3, \dots, m+1|2, \dots, m]$ can be obtained from $F(1, 1)$ by replacing its first row of $F(1, 1)$ by a row of all 1's and then cyclically permuting the rows, H has a nonzero term in its determinant expansion. Since the complementary submatrix of H in B has two oppositely signed nonzero terms in its determinant expansion, we again contradict the sign-nonsingularity of B.

Without loss of generality we now assume that B has a -1 in its upper left corner and hence a 0 in row 2 below this -1. The matrix $F(2, 1)$ has a nonzero term in its determinant expansion and hence there is a column $k \geqslant 2$, say column 2, whose entry in row 1 is -1 such that the submatrix $K = B[3, \dots, m|3, \dots, m]$ of B has a nonzero term in its determinant expansion. The complementary submatrix of K in B is

$$B[1, 2, m+1|1, 2, m+1] = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

which has two oppositely signed nonzero terms in its determinant expansion. This again contradicts the sign-nonsingularity of B and proves that b contains only one 1. ∎

Let F be an SNS matrix of order m. Then the unit column vector e_i has the property that

$$\left[\begin{array}{c|c} F & e_i \end{array}\right]$$

is a totally L-matrix if and only if each of the submatrices $F(i, 1), \dots, F(i, m)$ is an SNS matrix. An SNS matrix is *maximal* provided replacing any nonzero entry with ± 1 results in a matrix which is not an SNS matrix. Let $F = [f_{ij}]$ be a maximal SNS matrix of order m. If $f_{ij} = 0$, then $F(i, j)$ is not an SNS matrix and indeed has nonzero terms in its determinant expansion of opposite sign. If $f_{ij} \neq 0$, then either $F(i, j)$ is an SNS-matrix or every term in its determinant expansion equals 0; if F is also fully indecomposable, then $F(i, j)$ is always an SNS matrix. Hence, if F is a fully indecomposable maximal SNS matrix of order m, then

$$\left[\begin{array}{c|c} F & e_i \end{array}\right]$$

is a m-by-$m + 1$ totally L-matrix if and only if row i of F contains no zeros. An example of a fully indecomposable maximal SNS matrix of order $m = 4$ which cannot be extended to a 4-by-5 totally L-matrix is

$$\begin{bmatrix} 0 & -1 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

is a 4-by-5 totally L-matrix having no unit column, which shows that the full indecomposability assumption in Theorem 5 cannot be dropped. This is an instance of the following general construction. Let

$$\left[\begin{array}{c|c} A_1 & b_1 \end{array}\right] \text{ and } \left[\begin{array}{c|c} A_2 & b_2 \end{array}\right]$$

be totally L-matrices of sizes m_1 by $m_1 + 1$ and m_2 by $m_2 + 1$, respectively. Then

$$\left[\begin{array}{c|c|c} A_1 & O & b_1 \\ \hline O & A_2 & b_2 \end{array}\right]$$

is an $m_1 + m_2$-by-$m_1 + m_2 + 1$ totally L-matrix.

THEOREM 6. *If A is an m-by-$m + k$ totally L-matrix with $m \geqslant 2$, then $k \leqslant 2$.*

Proof. We prove the theorem by induction on m. Suppose that $m = 2$. Up to multiplication by -1, there are exactly four nonzero $(0, 1, -1)$ column vectors of size 2. Hence a totally L-matrix with two rows has at most four columns. Now assume that $m \geqslant 3$. First suppose that A has a unit column. Then for some i and j, $A(i, j)$ is an $m - 1$-by-$m - 1 + k$ totally L-matrix and, therefore by induction, $k \leqslant 2$. Hence we may assume that A has no unit columns, and therefore by Theorem 5 no submatrix of A of order m is fully indecomposable. Thus each submatrix of A of order m contains an r-by-$m-r$ zero submatrix for some r with $1 \leqslant r \leqslant m - 1$. Suppose that no submatrix of A of order m contains an r-by-$m - r$ zero submatrix with $2 \leqslant r \leqslant m - 1$. Each row of A contains at most one 1-by-$m - 1$ zero submatrix, and this zero matrix is a submatrix of at most $k + 1$ submatrices of A of order m. Since A has $\binom{m+k}{m}$ submatrices of order m, we have

$$\binom{m+k}{m} \leqslant m(k+1),$$

implying $k \leqslant 2$. We now assume that A has an r-by-$m - r$ zero submatrix for some r with $2 \leqslant r \leqslant m - 1$. Hence there are permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A' & 0 \\ X & Y \end{bmatrix},$$

where A' is an r-by-$r+k$ totally L-matrix. Since $2 \leqslant r \leqslant m-1$, the inductive hypothesis implies that $k \leqslant 2$. ■

Since a $(0, 1, -1)$ matrix of order m is an SNS matrix if and only if each of its signings has at least one unisigned column, it follows that an m-by-$m+1$ $(0, 1, -1)$ matrix is a totally L-matrix if and only if each of its signings has at least two unisigned columns; and for an m-by-$m + 2$ totally L-matrix, there must be at least three unisigned columns for each signing.

We now describe a method, which starting from the 2-by-4 totally L-matrix T in (2), allows one to construct an m-by-$m + 2$ totally L-matrix for every $m \geqslant 2$. We then show that every m-by-$m+2$ totally L-matrix with $m \geqslant 2$ can be constructed by this method starting from T.

Let $B = [b_{ij}]$ be an m-by-$m + 2$ totally L-matrix. Suppose that column t of B is a unit column, $b_{ut} \neq 0$, and row u has exactly three nonzero entries, occuring in columns r, s, and t. Let $A = [a_{ij}]$ be the $m + 1$-by-$m + 3$ matrix obtained from B by bordering on the right by a column of 0's and then on the bottom by the row vector all of whose entries are 0 except for a 1 in columns r and $m+3$ and the value $-b_{ur}b_{ut}$ in column t (this value is chosen so that $A[\{u, m+1\}, \{r, t\}]$ is an SNS matrix). The matrix A is called the *single extension* of B on columns t and r. Let $C = [c_{ij}]$ be the $(m + 2)$-by-$(m + 4)$

matrix obtained from A by bordering on the right by a column of 0's and then on the bottom by the row vector all of whose entries are 0 except for a 1 in columns s and $m+4$ and the value $-b_{us}b_{ut}$ in column t (this value is chosen so that $A[\{u, m+1, m+2\}, \{r, s, t\}]$ is an SNS matrix). The matrix C is called the *double extension* of B on column t.

The single extension of T on columns 1 and 3 and the double extension of T on column 3 are, respectively,

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix}. \tag{6}$$

LEMMA 7. *Let A be the single extension of B on columns t and r. Then A is a totally L-matrix.*

Proof. It suffices to show that each signing DA of A has at least three unisigned columns. Let $D = D_1 \oplus [d_{m+1}]$ be of order $m+1$ where $D_1 = O$ or D_1 is a signing of order m. If $d_{m+1} = 0$, then D_1 is a signing, D_1B has at least three unisigned columns, and the corresponding columns of DA are unisigned. If $D_1 = O$, then $d_{m+1} \neq 0$ and DA has three unisigned columns. Now suppose that $d_{m+1} \neq 0$ and $D_1 \neq O$. If D_1B has two unisigned columns neither of which is column r or column t, then the corresponding columns of DA, along with column $m+3$ of DA, are unisigned. Now assume that at most one of the columns of D_1B different from columns r and t is unisigned. Then D_1B has exactly three unisigned columns, two of which are columns r and t. Let q denote the other unisigned column of D_1B. Since $a_{m+1,t} = -b_{ur}b_{ut}$, and t is a unit column, either column r or column t of DA is unisigned. But columns q and $m+3$ of DA are also unisigned. Hence DA has at least three unisigned columns. We conclude that A is a totally L-matrix. ∎

LEMMA 8. *Let C be the double extension of B on column t. Then C is a totally L-matrix.*

Proof. Let $D = D_1 \oplus [d_{m+1}] \oplus [d_{m+2}]$ be a signing of order $m+2$ where $D_1 = O$ or D_1 is a signing of of order m. The matrices obtained from C by deleting row $m+1$ and column $m+3$ (respectively, row $m+2$ and column $m+4$) are single extensions of a totally L-matrix. Thus, if either $d_{m+1} = 0$ or $d_{m+2} = 0$, Lemma 7 implies that DC has at least three unisigned columns. We now suppose that both d_{m+1} and d_{m+2} are nonzero. Then columns $m+3$ and $m+4$ of DC are unisigned. If there is a unisigned column q of D_1B with $q \notin \{r, s, t\}$, then column q and columns $m+3$ and $m+4$ are three unisigned columns of DA. Now assume that columns r, s, and t are the only unisigned columns of D_1B. Since the matrix obtained from D_1B by multiplying its last row by -1 has at least three unisigned columns, columns r and s of DB are unit columns. It now follows that one of columns r, s and t of DC is unisigned, and hence that DC has at least three unisigned columns. Hence C is a totally L-matrix. ■

We note that if an m-by-$m+2$ totally L-matrix B can be singly extended to A and hence doubly extended to C, then B and C can also be extended. Hence by successive extensions of the matrix T in (2) we obtain m-by-$m+2$ totally L-matrices for each $m \geqslant 2$. We now show that every m-by-$m+2$ totally L-matrix can be obtained in this way.

LEMMA 9. *Let A be an m-by-$m+2$ totally L-matrix with $m \geqslant 2$. Then A has at least two unit columns and each row has exactly three nonzero entries.*

Proof. We prove the lemma by induction on m. If $m = 2$, then up to column permutations and multiplication of columns by -1, A is the matrix T of (2). Hence the lemma holds if $m = 2$. Now suppose that $m > 2$.

We first show that A has at least one unit column. If A has a fully indecomposable submatrix of order m, then by Theorem 5 A has at least two unit columns. We now assume that each submatrix of A of order m is not fully indecomposable. After row and column permutations we may assume that A has the form

$$A = \begin{bmatrix} X & O \\ Y & Z \end{bmatrix}$$

where X is p by $p+2$, Z is q by q, $p+q = m$, and q is minimal. It follows that X is a totally L-matrix and Z is a fully indecomposable SNS matrix. Suppose to the contrary that $q > 1$. By Theorems 1 and 3 there exists an strict signing D such that DZ has a unique unisigned column. Without loss of generality we assume that this column is $(+,0)$-unisigned. Let $v = (v_1, \ldots, v_{p+2})$, where

$$v_i = \begin{cases} 1 & \text{if column } i \text{ of } DY \text{ is } (+,0)\text{-unisigned,} \\ -1 & \text{if column } i \text{ of } DY \text{ is } (-,0)\text{-unisigned,} \\ 1 & \text{if column } i \text{ of } DY \text{ is nonzero and balanced,} \\ 0 & \text{if column } i \text{ of } DY \text{ is a zero column.} \end{cases}$$

Consider the matrix

$$M = \begin{bmatrix} X & O \\ v & 1 \end{bmatrix}.$$

Because M is $p+1$ by $p+3$, M is a totally L-matrix if and only if for every signing E, EM has at least three unisigned columns. Let $E = \operatorname{diag}(e_1, \ldots, e_{p+1})$ be a signing. Let $U = \operatorname{diag}(e_1, \ldots, e_p) \oplus e_{p+1}D$. Because A is a totally L-matrix, UA has at least three unisigned columns. If $e_{p+1} = 0$, then since each unisigned column of UA is one of columns $1, 2, \ldots, p+2$ and since the corresponding columns of EM are unisigned, EM has at least three unisigned columns. If $e_{p+1} \neq 0$, then $e_{p+1}DZ$ has a unique unisigned column and the last column of EM is unisigned. For $i = 1, 2, \ldots, p+2$, if column i of UA is unisigned then column i of EM is also unisigned. Hence EM has at least three unisigned columns and it follows that M is a totally L-matrix. By the induction assumption, M has at least two unit columns. Since D is a strict signing, it follows from the definition of v that A has at least one unit column, contradicting $q > 1$. Hence $q = 1$ and A has at least one unit column.

Since Y is 1 by $m+1$ we write $Y = (y_1, \ldots, y_{m+1})$. Without loss of generality we assume that Y is a (0, 1) vector and $Z = [1]$. Let $w = (w_1, \ldots, w_{m+1})$ be a (0, 1)-vector obtained from Y by replacing all but two of its 1's with 0's. Consider the matrix

$$N = \begin{bmatrix} X & O \\ w & 1 \end{bmatrix}.$$

We claim that N is a totally L-matrix. Since A is a totally L-matrix, it suffices to show that every submatrix of N of order m has a nonzero term in its determinant expansion. By the Frobenius-König theorem this is equivalent to showing that N does not have an a-by-b zero submatrix with $a + b = m + 1$. Let $\mathbb{O}$ be an a by b zero submatrix of N. If $\mathbb{O}$ does not intersect the last row of N, then $\mathbb{O}$ is a submatrix of A, and hence $a + b \leqslant m$ (since A is a totally L-matrix). Suppose $\mathbb{O}$ intersects the last row of N. Then $b \leqslant m - 1$ (since the last row of N contains three 1's). If $a \geqslant 2$, then by deleting the last row of $\mathbb{O}$ and attaching a column of 0's (from the last column of N), we obtain an $(a-1)$ by $(b+1)$ zero submatrix of A, and hence $a+b = (a-1)+(b+1) \leqslant m$. If $a = 1$, then $a + b \leqslant m$. It follows that N is a totally L-matrix.

We now claim that A has two unit columns. First note that by the induction assumption, X has at least two unit columns. We may assume that

$$X = \begin{bmatrix} X_1 & O & O \\ X_2 & 1 & 0 \\ X_3 & 0 & 1 \end{bmatrix}.$$

Suppose that A does not have two unit columns. Then $y_m = y_{m+1} = 1$. Choosing w so that $w_m = w_{m+1} = 1$ we obtain the totally L-matrix

$$N = \begin{bmatrix} X_1 & O & O & O \\ X_2 & 1 & 0 & 0 \\ X_3 & 0 & 1 & 0 \\ O & 1 & 1 & 1 \end{bmatrix}$$

of size m by $m+2$. Consider the matrix X' obtained from X by replacing the 0 in position $(m-1, m)$ by -1. By induction, each row of X has exactly three nonzero entries. Thus X' has a row with four nonzero entries. By induction, X' is not a totally L-matrix. Thus there exists a signing D such that DX' has at most two unisigned columns. But DX has at least three unisigned columns. It follows that the last two columns of DX are both $(+, 0)$-unisigned or are both $(-, 0)$-unisigned, and exactly one other column of DX is unisigned. But now either $(D \oplus [1])N$ or $(D \oplus [-1])N$ has only two unisigned columns. This contradicts the fact that N is a totally L-matrix. We conclude that A has at least two unit columns.

Without loss of generality,

$$A = \begin{bmatrix} A_1 & O & O \\ A_2 & 1 & 0 \\ A_3 & 0 & 1 \end{bmatrix}. \tag{7}$$

Then

$$\begin{bmatrix} A_1 & O \\ A_2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_1 & O \\ A_3 & 1 \end{bmatrix}$$

are totally L-matrices. It follows from the inductive assumption that each row of A has exactly three nonzero entries. ∎

COROLLARY 10. *Let A be an m-by-$m+2$ totally L-matrix with $m \geqslant 2$. Let u be any row of A. Then there exists a row v of A such that up to column permutations and multiplication of columns by -1's, the 2-by-$m+2$ submatrix of A determined by rows u and v has the form*

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Hence every 3-by-2 submatrix of A has a zero entry.

Proof. The first conclusion follows by induction using Lemma 9 (see the end of its proof). Let D be a signing with exactly two nonzero diagonal entries. Then DA has at least three unisigned columns. It follows that no two rows of A have their three nonzero entries in the same three columns, and that the product of the entries of each 2-by-2 submatrix of A equals 0 or -1. If there were a 3-by-2 submatrix of A with no zero entries, then the square of the product of its entries would equal $(-1)^3$. ∎

THEOREM 11. *Let A be an m-by-$m+2$ totally L-matrix with $m \geqslant 2$. Then up to row and column permutations and multiplication of rows and columns by -1, A either equals T or is obtained from T by a sequence of single and double extensions.*

Proof. We prove the theorem by induction on m. If $m = 2$, the theorem holds. Now assume that $m > 2$. By Lemma 9, A has at least two unit columns. Without loss of generality we assume that

$$A = \begin{bmatrix} B & O \\ x & 1 \end{bmatrix}$$

where B is a totally L-matrix of size $m-1$ by $m+1$. If $m = 3$ then it follows from Lemma 9 and Corollary 10 that up to row and column permutations and multiplication of rows and columns by -1, A is a single extension of T.

Now assume that $m \geqslant 4$. By induction B is either a single extension or a double extension of a totally L-matrix C. First assume that B is a single extension of C. Without loss of generality we may assume that column m of C is a unit column whose nonzero entry is in row $m-2$, the nonzero entries in row $m-2$ of C are in columns $m-2$, $m-1$, and m, and B is the single extension of C in columns $m-1$ and m. First suppose that $a_{m,m} = 0$. If $a_{m,m+1} \neq 0$, then using Corollary 10 we see that A is a single extension of B on columns $m-2$ and $m+1$. If $a_{m,m+1} = 0$, then interchanging the last two rows of A results in a matrix which is the single extension of a totally L-matrix. Now suppose that $a_{m,m} \neq 0$. By Corollary 10 either $a_{m,m-2} \neq 0$ or $a_{m,m+1} \neq 0$. In the former case A is the double extension of C on columns m, and in the latter case A is the single extension of B on columns m and $m+1$. We conclude that if B is a single extension of C, then A is a single or double extension of a totally L-matrix.

Now suppose that B is a double extension of C. Without loss of generality, we assume that B is the double extension of C on column $m-1$, the nonzero entries in row $m-3$ of C are in columns $m-3$, $m-2$, and $m-1$, the nonzero entries in row $m-2$ of A are in columns $m-2$, $m-1$, and m, and the nonzero entries in row $m-1$ of A are in columns $m-3$, $m-1$, and $m+1$. If $a_{m,m-1} = 0$, then by Corollary 10 one of $a_{m,m}$ and $a_{m,m+1}$ is 0 and we may permute rows $m-2$, $m-1$ and m to obtain a matrix which is the double extension on column $m-1$ of a totally L-matrix. Suppose $a_{m,m-1} \neq 0$. By Corollary 10 the other nonzero entry of row m lies in either column m or column $m+1$. In either case, A is easily seen to be a single extension of a totally L-matrix. ■

A *strong L-matrix* is an L-matrix such that every square submatrix is either an SNS matrix or has the property that each term in its determinant expansion equals 0. Let A be an m-by-$m+2$ totally L-matrix which is obtained from the 2-by-4 matrix T by a sequence of single extensions. We describe an alternative method for constructing A starting from the matrix $I_1 = [1]$ of order 1. We then show that A is a strong L-matrix.

Let $R = [r_{ij}]$ be a matrix of order k. Assume $r_{ij} \neq 0$. Let R' be the matrix of order $k+1$ obtained from R by bordering on the right by the unit column vector with $-r_{ij}$ in position i and on the bottom by the row vector with 1's in positions j and $k+1$ and 0's elsewhere. We call R' the *expansion* of R on r_{ij}.

THEOREM 12. *If R is a strong SNS matrix, then R' is also a strong SNS matrix. In particular, a matrix obtained from I_1 by a sequence of expansions on nonzero entries is a strong SNS matrix.*

Proof. Suppose that R is a strong SNS matrix of order n. It follows from Theorems 3.4 and 3.5 of [3] that an expansion of an SNS matrix is an SNS matrix. In particular we know that R' is an SNS matrix. Let X be a square submatrix of R'. If X does not contain the submatrix $R'[i, n+1 | j, n+1]$, then clearly either X is an SNS matrix or every term in its determinant expansion equals 0. Otherwise, X is an expansion of a square submatrix S of R on r_{ij}. If S is an SNS matrix, then so is X. If every term in the determinant expansion of S equals 0, then X is an SNS matrix or all terms in its determinant expansion are 0 according to which of these properties $X(i, j)$ has. ■

The matrix

$$K = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

is obtained from I_1 by an expansion. Bordering this matrix by two unit columns, we obtain T. If, starting with K, we do a sequence of expansions, always expanding on a nonzero entry whose row contains only two nonzero entries, we obtain a strong SNS matrix with two rows having exactly two nonzero entries and the remaining rows having exactly three nonzero entries. Bordering this matrix by two unit columns we get a matrix which is obtained from T by a sequence of single extensions. Conversely, every matrix obtained from T by a sequence of single extensions arises in this way. It follows from Theorem 12 that matrices obtained from T by a sequence of single extensions are strong L-matrices. The matrix on the right in (6) is obtained from T by a double extension but is not a strong L-matrix.

5. BARELY L-MATRICES

Let A be an m-by-n L-matrix. We recall from Section 3 that $\mathcal{A}_i$ denotes the set of signings D such that column i is the unique unisigned column of DA, and that A is a barely L-matrix if and only if each $\mathcal{A}_i$ is nonempty. If A is an L-indecomposable barely L-matrix, then by Theorem 1 each $\mathcal{A}_i$ contains a strict signing.

Suppose that

$$A = \begin{bmatrix} U & O \\ V & W \end{bmatrix}. \tag{8}$$

If U and W are L-matrices and A is a barely L-matrix, then it follows by definition that U and W are also barely L-matrices. Hence the matrices in a set of L-indecomposable components of a barely L-matrix are also barely L-matrices. The converse also holds. Its proof makes use of the following lemma.

LEMMA 13. *Let A be an L-matrix of the form* (8). *If U is a barely L-matrix, then W is an L-matrix and hence* (8) *is an L-decomposition of A.*

Proof. The lemma follows by induction on the number of rows once we prove it under the additional assumption that U is L-indecomposable. Assume U is an L-indecomposable barely L-matrix. Suppose to the contrary that W is not an L-matrix. Then there is a balanced signing FW of W. Since A is an L-matrix, for an appropriate size zero matrix O, $(O \oplus F)A$ has at least one unisigned column and hence FV has at least one unisigned column, say column i. We may assume that column i of FV is $(+, 0)$-unisigned. Since U is assumed to be L-indecomposable, it follows from Theorem 1 that $\mathcal{A}_i$ contains a strict signing. Let E be a strict signing such that column i of EU is $(-, 0)$ unisigned and every other column of EU is balanced. Since E is a strict signing and U can have no zero columns, every column of EU other than column i contains both a 1 and -1. We now conclude that $E \oplus F$ is a balanced signing of A, contradicting the fact that A is an L-matrix. ■

COROLLARY 14. *Let A be a* $(0, 1, -1)$ *matrix of the form* (8). *If U and W are barely L-matrices, then A is a barely L-matrix.*

Proof. Assume that U and W are barely L-matrices. Then A is an L-matrix. We prove that A is a barely L-matrix by showing that for each i, the matrix $A(i)$ obtained from A by deleting column i is not an L-matrix. This is clear if column i intersects U. If column i intersects W, then by Lemma 13 $A(i)$ is not an L-matrix. ■

The next corollary is a consequence of Corollary 14 and the remarks at the beginning of this section.

COROLLARY 15. *Let A be an L-matrix, and let $\{A_1, A_2, \ldots, A_k\}$ be a set of L-indecomposable components of A. Then A is a barely L-matrix if and only if each of the matrices A_i is a barely L-matrix.*

Recall that an m-by-n L-matrix A is extremal provided A is not a submatrix of an $m+1$-by-n L-matrix.

THEOREM 16. *Let A be a* $(0, 1, -1)$ *matrix. Then A is a barely L-matrix if and only if A is an extremal L-matrix.*

Proof. First suppose that A is not a barely L-matrix. If A is not an L-matrix, then A is not an extremal L-matrix. If A is an L-matrix, then $A(i)$ is an L-matrix for some i, and the matrix obtained from A by attaching the unit row with a 1 in the ith position is an L-matrix. Hence in either case A is not an extremal L-matrix.

Now suppose that A is a barely L-matrix but A is not an extremal L-matrix. There exists a row vector x such that

$$\left[\frac{A}{x}\right]$$

is an L-matrix. Appending a column of 0's to this matrix, we contradict Lemma 13. ■

THEOREM 17. *Up to row and column permutations, a barely L-matrix has a unique set of L-indecomposable components.*

Proof. Let A be an m-by-n barely L-matrix. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be an ordered partition of $\{1, 2, \ldots, m\}$ and let $\beta = (\beta_1, \ldots, \beta_k)$ be an ordered partition of $\{1, 2, \ldots, n\}$ such that $A_i = A[\alpha_i|\beta_i]$ $(1 \leqslant i \leqslant k)$ is an L-indecomposable barely L-matrix and $A[\alpha_i|\beta_j] = O$ $(1 \leqslant i < j \leqslant k)$. Also, let $\gamma = (\gamma_1, \ldots, \gamma_l)$ be an ordered partition of $\{1, 2, \ldots, m\}$, and let $\delta = (\delta_1, \ldots, \delta_l)$ be an ordered partition of $\{1, 2, \ldots, n\}$ such that $B_i = A[\gamma_i|\delta_i]$ $(1 \leqslant i \leqslant l)$ is an L-indecomposable barely L-matrix and $A[\gamma_i|\delta_j] = O$ $(1 \leqslant i < j \leqslant l)$. We prove the theorem by showing that $k = l$, and that γ is a permutation of α and δ is the same permutation of β.

Let p be the smallest integer such that $\alpha_p \cap \gamma_l \neq \emptyset$. Let q be the smallest integer such that $\beta_q \cap \delta_l \neq \emptyset$. If $p < q$, then B_l has a row of all 0's, contradicting the fact that B_l is an L-matrix. If $q < p$, then A_q has a column of all 0's, contradicting the fact that A_q is a barely L-matrix. Hence $p = q$ and $A[\alpha_p \cap \gamma_l|\beta_p \cap \delta_l]$ is a nonvacuous matrix. It follows from the minimality of p that $A[\alpha_p \cap \gamma_l|\delta_l \setminus (\beta_p \cap \delta_l)] = O$. Since B_l is an L-matrix, it follows that $A[\alpha_p \cap \gamma_l|\beta_p \cap \delta_l]$ is an L-matrix.

First suppose that $\alpha_p \cap \gamma_l \neq \alpha_p$ and $\beta_p \cap \delta_l \neq \beta_p$. Since $A[\gamma_1 \cup \cdots \cup \gamma_{l-1}|\delta_l] = O$, it follows that $A[\alpha_p \setminus \gamma_l|\beta_p \cap \delta_l] = O$. Since A_p is an L-matrix, $A[\alpha_p \setminus \gamma_l|\beta_p \setminus \delta_l]$ is an L-matrix. We now have an L-decomposition of A_p, contradicting the L-indecomposability of A_p. Thus either (i) $\alpha_p \cap \gamma_l = \alpha_p$ or (ii) $\beta_p \cap \delta_l = \beta_p$.

In fact we now claim that both (i) and (ii) hold. First suppose that (i) but not (ii) holds. Then since B_l is an L-matrix, some matrix obtained from A_p by deleting a column is an L-matrix, contradicting the fact that A_p is a barely

L-matrix. If (ii) but not (i) holds, then A_p has a zero row, contradicting the fact that A_p is an L-matrix. Thus $\alpha_p \subseteq \gamma_l$ and $\beta_p \subseteq \delta_l$, and hence

$$B_l = \begin{bmatrix} A_p & O \\ X & Y \end{bmatrix}$$

for some X and Y. If Y is nonvacuous, then since A_p is a barely L-matrix, it follows from Lemma 13 that Y is an L-matrix, contradicting the fact that B_l is an L-indecomposable matrix. If Y has columns but not rows, then B_l is a barely L-matrix with at least one zero column, a contradiction. If Y has rows but not columns, then A_p is a barely L-matrix which is not extremal, contradicting Theorem 16. It follows that $\gamma_l = \alpha_p$ and $\delta_l = \beta_p$. Deleting the rows in γ_l and the columns in δ_l from A, the theorem follows by induction.■

A special case of Theorem 17 is that the L-indecomposable components of an SNS matrix are unique up to row and column permutations. It follows from Theorem 2 that the L-indecomposable components are fully indecomposable. Hence Theorem 17 implies that the fully indecomposable components of a square matrix are unique up to row and column permutations if the matrix has the zero-nonzero pattern of an SNS matrix. It is well known that the same conclusion holds for any matrix which has a nonzero term in its determinant expansion [2].

Let $m \geqslant 1$. We partition the subsets of $\{1, 2, \dots, m\}$ into pairs consisting of a set and its complement, and we choose one set F from each pair. Let $\hat{F}$ denote the column m-tuple which has a 1 in the ith row if $i \in F$ and a -1 if $i \notin F$. We call an m-by-2^{m-1} matrix whose columns are the $\hat{F}$'s in some order an *S_m-matrix*. Let A be an S_m-matrix. For each signing D there is at least one unisigned column of DA, and when the signing is strict there is exactly one unisigned column of DA. Hence A is a barely L-matrix. An S_m-matrix contains no 0's and hence is L-indecomposable. Moreover, if $m > 1$, the largest order of an SNS submatrix of an S_m-matrix is 2. Every m-by-n L-matrix with no 0's contains an S_m-matrix as a submatrix. This is because every strict signing must unisign as least one of its columns.

THEOREM 18. *Let A be an m by n barely L-matrix. Then $n \leqslant 2^{m-1}$. If $m \geqslant 3$, then equality holds if and only if A is an S_m-matrix.*

Proof. First assume that A is L-indecomposable. By Theorem 1 each $\mathcal{A}_i$ contains a strict signing D_i. Since $-D_i$ is also a strict signing in $\mathcal{A}_i$ and since there are exactly 2^m strict signings of A, it follows that $n \leqslant 2^{m-1}$. Assume that $n = 2^{m-1}$. Then each strict signing belongs to some $\mathcal{A}_i$ and hence unisigns exactly one column of A. Therefore for each i, D_i and $-D_i$ are the only strict signings which unisign column i. Suppose that column j of A contains a 0, say in row k. Then $m \geqslant 2$, and replacing the kth diagonal entry of D_j with its negative gives a strict signing $E \neq D_j, -D_j$ which unisigns column j. Hence A has no 0's, and it follows that A is an S_m-matrix.

Now assume that A is L-decomposable, and let $A_1, A_2, \ldots, A_k$ be its L-indecomposable components, where $k \geqslant 2$. Let m_i be the number of rows of A_i $(i = 1, 2, \ldots, k)$. Since the A_i are barely L-matrices, using what we have proved above, we now obtain

$$n \leqslant 2^{m_1-1} + 2^{m_2-1} + \cdots + 2^{m_k-1} \leqslant 2^{m-1}.$$

If $m \geqslant 3$ we have $n < 2^{m-1}$. The theorem now follows. ■

COROLLARY 19. *Let m be a positive integer. Then there exists an m-by-n barely L-matrix if and only if $m \leqslant n \leqslant 2^{m-1}$.*

Proof. By Theorem 18 it suffices to show that there exists an m-by-n barely L-matrix for all n in the indicated range. This is obvious for $m = 1$. We proceed inductively from m to $m+1$. If $m + 1 \leqslant n \leqslant 2^{m-1} + 1$, then there exists an m-by-$n-1$ barely L-matrix A and hence $A \oplus I_1$ is a $m+1$-by-n barely L-matrix. Now suppose that $2^{m-1} + 1 \leqslant n \leqslant 2^m$ and suppose that $n = 2^{m-1} + t$. Let

$$\begin{bmatrix} A_1 & A_2 \end{bmatrix}$$

be an S_m-matrix where A_2 has t columns. Let

$$B = \left[\begin{array}{ccc|ccc|ccc} & A_1 & & & A_2 & & & A_2 & \\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 & -1 & \cdots & -1 \end{array}\right].$$

Then it is easy to verify that every signing of B has a unisigned column and every column of B is the only unisigned column of some signing. Hence B is an $m+1$-by-n barely L-matrix. ■

We now obtain a bound on the number of columns of a barely L-matrix without any -1's. We will make use of the following special case of a theorem of Milner [6]. Let $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ be a family of subsets of $\{1, 2, \ldots, m\}$. Then $\mathcal{X}$ is called an *intersecting family* provided $X_i \cap X_j \neq \emptyset$ for all $i \neq j$. The family $\mathcal{X}$ is an *antichain* provided $X_i \not\subseteq X_j$ for all $i \neq j$.

THEOREM 20. *If $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ is an intersecting antichain of subsets of $\{1, 2, \ldots, m\}$, then*

$$n \leqslant \binom{m}{\lceil \frac{m+1}{2} \rceil}. \tag{9}$$

Equality holds in (9) if and only if one of the following holds:

(a) *$\mathscr{X}$ is the family of all subsets of cardinality $\lceil (m+1)/2 \rceil$ of $\{1, 2, \ldots, m\}$.*
(b) *m is even and there exists an integer $k \in \{1, 2, \ldots, m\}$ such that $\mathscr{X}$ consists of all subsets of cardinality $m/2$ of $\{1, 2, \ldots, m\}$ containing k and all subsets of cardinality $m/2 + 1$ not containing k.*

Let $\mathscr{X} = \{X_1, X_2, \ldots, X_n\}$ be a type (a) or a type (b) family. We note for later use that if U is a proper subset of X_j then there exists an $i \neq j$ such that U is a proper subset of X_i.

If $\mathscr{X} = \{X_1, X_2, \ldots, X_n\}$ is a family of subsets of $\{1, 2, \ldots, m\}$, then the *incidence matrix* of $\mathscr{X}$ is the m-by-n (0,1) matrix $A = [a_{ij}]$ such that $a_{ij} = 1$ if and only if $i \in X_j$.

LEMMA 21. *Let A be the incidence matrix of the family $\mathscr{X}$ of all subsets of cardinality $k > 0$ of $\{1, 2, \ldots, m\}$. Then A is an L-matrix if and only if $1 \leqslant k \leqslant \lfloor (m+1)/2 \rfloor$. The matrix A is a barely L-matrix if and only if $k = 1$, or m is odd and $k = (m+1)/2$.*

Proof. First suppose that $k > \lfloor (m+1)/2 \rfloor$. Then for any strict signing D with $\lfloor (m+1)/2 \rfloor$ positive ones and $\lceil (m-1)/2 \rceil$ negative ones, DA is a balanced signing of A. Hence A is not an L-matrix. Now suppose that $k \leqslant \lfloor (m+1)/2 \rfloor$. In each signing of order m there is a set of $\lfloor (m+1)/2 \rfloor$ diagonal entries which either are nonnegative and not all 0 or are nonpositive and not all 0. Hence each signing unisigns some column of A, and A is an L-matrix.

If $k = 1$, then A is a permutation matrix of order m and hence A is a barely L-matrix. Next suppose that m is odd and $k = (m+1)/2$. Let F be any subset of $\{1, 2, \ldots, m\}$ of cardinality $(m+1)/2$. Let D_F be the strict signing where the ith diagonal entry equals 1 if and only if $i \in F$. Then since $\mathscr{X}$ is an intersecting antichain, the column of $D_F A$ corresponding to F is the unique unisigned column of $D_F A$. We conclude that A is a barely L-matrix. Finally, suppose that $2 \leqslant k \leqslant \lfloor m/2 \rfloor$. Let D be a signing which $(+, 0)$-unisigns column i of A. If there is a p such that the pth entry of column i equals 0 and the pth diagonal entry of D equals 0 or 1, then since $k \geqslant 2$, some column different from column i of A is also $(+, 0)$-unisigned by D. Otherwise, since $k \leqslant \lfloor m/2 \rfloor$, D has at least $\lceil m/2 \rceil$ diagonal entries equal to -1 and some column of A is $(-, 0)$-unisigned by D. It follows that every m-by-$n - 1$ submatrix of A is an L-matrix, and in particular that A is not a barely L-matrix. ■

LEMMA 22. *The incidence matrix of a type (b) family with $m > 2$ is a barely L-matrix.*

Proof. Let A be the incidence matrix of a type (b) family $\mathscr{X}$. Let D be a signing of order m. If the kth diagonal entry of D equals 0, then some column of DA corresponding to a set of $\mathscr{X}$ containing k is unisigned. Now suppose the kth diagonal entry of D equals 1. If D contains at most $m/2$ negative ones, then some column of DA corresponding to a set containing k is unisigned. Otherwise some column of DA corresponding to a set of $\mathscr{X}$ not containing k is unisigned. It follows that A is an L-matrix. Since $\mathscr{X}$ is an intersecting antichain, it follows as in the proof of Lemma 21 that A is a barely L-matrix. ■

THEOREM 23. *Let A be an m-by-n barely L-matrix of 0's and 1's with $m > 2$. Then*

$$n \leqslant \binom{m}{\lceil \frac{m+1}{2} \rceil}. \tag{10}$$

If $m \geqslant 5$, then equality holds in (10) if and only if m is odd and A is the incidence matrix of a type (a) family of subsets of $\{1, 2, \ldots, m\}$, or m is even and A is the incidence matrix of a type (b) family.

Proof. The matrix A is the incidence matrix of a family $\mathscr{X} = \{X_1, X_2, \ldots, X_n\}$ of n subsets of $\{1, 2, \ldots, m\}$. First assume that A is L-indecomposable. By Theorem 1 each $\mathscr{A}_i$ contains a strict signing D_i. Without loss of generality we may assume that column i of D_iA is $(+, 0)$-unisigned. Let Y_i be the subset of $\{1, 2, \ldots, m\}$ consisting of those integers j such that the jth diagonal entry of D_i equals 1. Let $\mathscr{Y} = \{Y_1, Y_2, \ldots, Y_n\}$. Let i and j be integers with $1 \leqslant i < j \leqslant n$. Since column j of D_iA is balanced, $X_j \cap Y_i \neq \emptyset$ and $X_j \cap \overline{Y_i} \neq \emptyset$. Since D_j unisigns column j of A, it follows that $X_j \subseteq Y_j$. A similar argument shows that $X_i \subseteq Y_i$. Hence $Y_j \cap Y_i \neq \emptyset$ and $Y_j \cap \overline{Y_i} \neq \emptyset$, and $\mathscr{Y}$ is an intersecting antichain. It follows from Theorem 20 that (10) holds. Suppose equality holds in (10). By Theorem 20 $\mathscr{Y}$ is either a type (a) family or a type (b) family. Suppose that some X_j is a proper subset of Y_j. Then there exists $i \neq j$ such that X_j is a subset of Y_i. This implies that column j of D_iA is unisigned contradicting the fact that column i is the unique unisigned column of D_iA. Hence $X_j = Y_j$ for all $j = 1, 2, \ldots, n$ and $\mathscr{X}$ is a type (a) or (b) family. If $\mathscr{X}$ is a type (a) family, then by Lemma 21 m is odd.

Now assume that A is L-decomposable, and let $A_1, A_2, \ldots, A_k$ be its L-indecomposable components where $k \geqslant 2$. Let m_i be the number of rows of A_i $(i = 1, 2, \ldots, k)$. It is easy to verify that every barely L-matrix of 0's and 1's with two rows is L–decomposable, and that I_1 is the only barely L-matrix of 0's and 1's with one row. Let m_i be the L-matrices, using what we have proved above, we now obtain

$$n \leqslant \binom{m_1}{\lceil \frac{m_1+1}{2} \rceil} + \cdots + \binom{m_k}{\lceil \frac{m_k+1}{2} \rceil} \leqslant \binom{m}{\lceil \frac{m+1}{2} \rceil}.$$

If $m \geqslant 5$ we have

$$n < \binom{m}{\lceil \frac{m+1}{2} \rceil}.$$

Since by Lemmas 21 and 22 the incidence matrix of a type (a) family with m odd and that of a type (b) family are barely L-matrices, the theorem now follows. ∎

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